Ranges of Bimodule Projections and Conditional Expectations

Ranges of Bimodule Projections and Conditional Expectations

By

Robert Pluta

CAMBRIDGE SCHOLARS
PUBLISHING

Ranges of Bimodule Projections and Conditional Expectations,
by Robert Pluta

This book first published 2013

Cambridge Scholars Publishing

12 Back Chapman Street, Newcastle upon Tyne, NE6 2XX, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-4438-4612-0, ISBN (13): 978-1-4438-4612-7

Contents

Acknowledgements

This book is a revised version of my Ph. D. thesis of the same title completed in 2012 at Trinity College Dublin under the supervision of Professor Richard M. Timoney to whom I would like to express my sincere gratitude. All work contained in this book reflects his insights and none of it would have been possible without his guidance.

My heartfelt thanks go to my parents who have always encouraged me to pursue my academic interests. It is to them that this book is dedicated.

Robert Pluta
University of Iowa
February 2013

Chapter 1

Introduction

In ring theory and in the context of algebras, idempotents have well-established uses. In particular, if $e \in R$ is an idempotent of a ring R, then the subring eRe has unit e and there is an eRe-bimodule projection $x \mapsto exe$ from R onto eRe. The kernel $eR(1-e) + (1-e)Re + (1-e)R(1-e)$ of the projection is a complementary eRe-submodule of R.

In probability theory and in the theory of von Neumann algebras the notion of conditional expectation (as a completely positive map $\mathcal{E}\colon M \to M$ on a von Neumann algebra M, with M commutative in the case of probability theory) satisfies similar algebraic properties as the Peirce projections on a ring R or on an algebra A. A result of J. Tomiyama [Tomiyama, 1957] states that a unital and bounded projection $\mathcal{E}\colon A \to A$ with range $S = \mathcal{E}(A)$ a C^*-subalgebra of A must have norm one, must be positive, must satisfy the conditional expectation property $\mathcal{E}(s_1 x s_2) = s_1 \mathcal{E}(x) s_2$ (for $s_1, s_2 \in S$, $x \in A$) and also the Schwarz type inequality $(\mathcal{E}(x))^* \mathcal{E}(x) \leq \mathcal{E}(x^* x)$. In one of the themes of recent research, the notion of injective operator space, a similar algebraic 'conditional expectation' property plays a significant role, interacting with the notion of a ternary ring of operators (TRO).

In [Lam, 2006] T. Y. Lam proposed abstracting the algebraic properties of the Peirce projection $\mathcal{E}_e \colon R \to R$ associated with an idempotent e in a ring R, which is given by $\mathcal{E}_e(x) = exe$ $(x \in R)$, and investigating algebraic properties that hold in this more general context. His proposal is to consider (additive) maps $\mathcal{E} \colon R \to R$ with $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$, $S = \mathcal{E}(R)$ a subring of R under the assumption that $\mathcal{E}$ is an S-bimodule map (which means that it satisfies the conditional expectation property $\mathcal{E}(s_1 x s_2) = s_1 \mathcal{E}(x) s_2$ for $s_1, s_2 \in S$, $x \in R$). Lam refers to such subrings S as 'corners'.

We consider this notion principally in the context of a (complex) Banach or C^*-algebra A in place of a ring R and with the assumption that the corner $S = \mathcal{E}(A)$ is a complex subalgebra. Our aim is to characterise such corners as fully as we can, ideally by establishing that they are related to the ranges of the more well-known completely positive (unital) conditional expectations.

Each chapter below begins with a summary of the results in it, and here is an overview of the structure of the chapters.

Chapter 2 outlines the general approach of Lam (in the context of rings) and includes some general facts that will be useful to us later on in the context of algebras. For instance, although a ring-theoretic Lam corner S of a unital algebra A need not be a subalgebra, if S is a subalgebra then $\mathcal{E}$ must be linear, which justifies the definition of corner algebra we use from Chapter 3 on (Definition 3.1.1). We also note that corner rings of unital rings must be inverse closed and point out that graphs of ring homomorphisms are corners, an idea we use later as a source of counterexamples. Chapter 2 also includes some results on properties inherited by corner rings (regularity, π-regularity and others).

In Chapter 3 we move to algebras, our main theme, and adopt a definition modified from the ring-theoretic one (which insists that we deal with corners that are subalgebras and have vector space complements, or equivalently we deal only with linear Lam conditional expectations $\mathcal{E}$). In fact we consider normed algebras, usually Banach algebras and often C^*-algebras A. We first observe that in general a Lam conditional expectation $\mathcal{E}\colon A \to A$ need not be continuous but the initial examples leave open the possibility that there always exists a continuous Lam conditional expectation $\tilde{\mathcal{E}}$ with the same range $S = \mathcal{E}(A) = \tilde{\mathcal{E}}(A)$. While simple examples show that Lam corners S in C^*-algebras need not be self-adjoint subalgebras, Peirce corners in C^*-algebras and certain 'generalised' Peirce corners are similar to self-adjoint corners. We characterise corners in $M_n(\mathbb{C})$ that contain the diagonal and use that to characterise self-adjoint corners of $\mathcal{B}(H)$ that contain the diagonal (in some basis for H). The latter result generalises to certain corners in purely atomic von Neumann algebras. A final result in this chapter is that if p is a projection in a primitive C^*-algebra, then A is prime if and only if the Peirce corner pAp is prime.

In Chapter 4 we propose a definition of a 'ternary' corner of a TRO T, which we consider as a generalisation of injective operator spaces. An injective operator space is completely isometric to a ternary corner pAq of an injective C^*-algebra A (with $p, q \in A$ projections). Hilbertian TROs are completely characterised as row or column Hilbert spaces and we establish that the ternary corners are precisely the closed subspaces in those cases.

Chapter 5 focusses largely on commutative C^*-algebras and we seek first to relate Lam conditional expectations to retracts on the spectrum of the algebra. Retractions $\tau\colon X \to X$ on a (locally compact) topological space X give rise to Lam conditional expectations $\mathcal{E}_\tau\colon C_0(X) \to C_0(X)$ via $\mathcal{E}_\tau(f) = f \circ \tau$. Such $\mathcal{E}_\tau$ maps have the additional property of being algebra *-homomorphisms. In fact we establish a converse, that a Lam conditional expectation $\mathcal{E}\colon C_0(X) \to C_0(X)$ which is also an algebra homomorphism must be given by a retraction of the one-point compactification $X^* = X \cup \{\omega\}$ of X fixing ω (Corollary 5.1.12). For commutative C^*-algebras we show that dense corners cannot be proper and that self-adjoint corners must be closed and always have closed complements (and may also have non-closed complements). Two final results in this chapter are that self-adjoint corners S of a C^*-algebra A are square-root closed, and if $x = x^* \in S$ then the C^*-subalgebra of A generated by x must be contained in S.

In Chapter 6 we first examine the concept of norming algebras and its application to the automatic complete boundedness problem. The basic issue here is to determine some algebraic conditions which ensure that a given bounded homomorphism between operator algebras is automatically completely bounded. In Theorem 6.1.5 we give one such condition. We then introduce AW^*-TROs which admit a type decomposition (type I, II, and III), and which generalize AW^*-algebras in a similar way as W^*-TROs generalize W^*-algebras.

Chapter 2

General Theory of Corner Rings

All rings to be consider are associative, but not necessarily unital.

In the first section of this chapter we will review the necessary background material from *Corner Ring Theory* [Lam, 2006]. In the succeeding sections, we will prove some new results, in particular:

- New proofs of uniqueness of complements (§2.2).
- Graphs of ring-homomorphisms are corner rings (§2.4). In particular, one can always find a Lam conditional expectation onto the graph of a ring homomorphism (between two rings); this expectation is multiplicative.
- If $\mathcal{E}\colon R \to R$ is an $\mathcal{E}(R)$-bimodule projection (or a Lam

conditional expectation), then $\mathcal{E}$ is multiplicative if and only if $\ker \mathcal{E}$ is an ideal (Proposition 2.4.2).

- Let A be a unital (associative) algebra over a field F, and S a corner of A regarded as a ring with complement M. If S is also a subalgebra of A then then the group M must be a vector space over F (Proposition 2.5.2).

- Every unital corner is inverse closed (Proposition 2.6.1).

- Hereditary properties of corners in regular and $*$-rings. In particular, every self-adjoint corner of a unital Baer $*$-ring is a Baer $*$-ring (Theorem 2.8.5).

2.1 Definitions and Main Theorems

The general theory of *corner rings* was first systematically established in T. Y. Lam's 2006 paper [Lam, 2006], but the concept – without explicit mention of the phrase 'corner ring' – had appeared earlier in the 1999 paper [Kraft et al., 1999] authored by H. Kraft, L. W. Small, and N. R. Wallach.

One of the main objectives of the theory of *corner rings* is to provide an axiomatic foundation for the classical theory of Peirce decompositions. Recall that the Peirce decomposition of an associative ring R with respect to an idempotent $e^2 = e \in R$ is

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e).$$

If R has no identity, $R(1-e) := \{x - xe : x \in R\}$ and $(1-e)R := \{x - ex : x \in R\}$. It is important to note that

$$S := eRe \quad \text{and} \quad M := eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$$

are respectively a subring of R and a subgroup of the additive group of R, and

$$R = S \oplus M, \quad SM \subseteq M, \quad MS \subseteq M.$$

This leads to the following definition of *corners* which allows one to study some features of Peirce decompositions in arbitrary rings, including those with no proper idempotents.

Definition 2.1.1 (Lam [Lam, 2006])**.** Let R be a ring. A subring S of R is called a *corner ring* (or simply a *corner*) of R if there exists a subgroup $M \subseteq R$ of the additive group of R such that

$$R = S \oplus M, \quad SM \subseteq M, \quad MS \subseteq M. \tag{2.1}$$

Such M is called a *complement* of S in R.

A useful characterization of corner rings, which was proved in [Lam, 2006], asserts that there is a one-to-one correspondence between corners of R and bimodular projections $\mathcal{E}\colon R \to R$ from R onto subrings of R. Specifically, this characterization is as follows.

Proposition 2.1.2 ([Lam, 2006, Proposition 2.1])**.** Let R be a ring, $S \subseteq R$ a subring. Then S is a corner ring of R if and only if there exists an S-bimodule map $\mathcal{E}\colon R \to R$ (which means that $\mathcal{E}$ is additive, and both left and right S-homogeneous, i.e., $\mathcal{E}(sx) = s\mathcal{E}(x)$ and $\mathcal{E}(xs) = \mathcal{E}(x)s$ for $s \in S$, $x \in R$) such that $\mathcal{E}(R) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. (We will refer to $\mathcal{E}$ as a *Lam conditional expectation.*)

Proof. If $\mathcal{E}\colon R \to R$ exists, one check easily that $M := \ker \mathcal{E}$ satisfies (2.1), so S is a corner of R. Conversely, if $S \subseteq R$ is a corner of R, choose some complement M so that $R = S \oplus M$, $SM \subseteq M$, $MS \subseteq M$. Then define $\mathcal{E}\colon R \to R$ by $\mathcal{E}(s+m) = s$ for $s \in S$ and $m \in M$. Clearly, $\mathcal{E}$ is additive. If $s_0 \in S$, we have

$$\mathcal{E}(s_0(s+m)) = \mathcal{E}(s_0 s + s_0 m) = s_0 s = s_0\mathcal{E}(s+m)$$

since $s_0 s \in S$ and $s_0 m \in M$. Thus, $\mathcal{E}$ is left S-homogeneous, and a similar check shows that $\mathcal{E}$ is also right S-homogeneous. It is straightforward to verify that $\mathcal{E}(R) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$.

The following proposition will be useful for further considerations and we will make reference to it many times. It states that if S is a corner of a unital ring R, then S always has an identity element, although this may not be the identity of R. In particular, if $\mathcal{E}\colon R \to R$ is an S-bimodule projection onto S as in Proposition 2.1.2, then $\mathcal{E}(1)$ is an identity of the ring S.

Proposition 2.1.3 ([Lam, 2006, Proposition 2.2])**.** Let R be a ring with unit 1, $S \subseteq R$ a corner of R with a complement

M so that $R = S \oplus M$, and let $1 = e + f$ where $e \in S$ and $f \in M$. Then e is an identity of the ring S. In particular, the decomposition $1 = e + f$ is independent of the choice of the complement M, e and f are complementary idempotents in R, and if $\mathcal{E} \colon R \to R$ is an S-bimodule map as in Proposition 2.1.2 (with $\mathcal{E}(R) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$) then $\mathcal{E}(1) = e$.

Proof. For any $s \in S$, $s = s1 = s(e + f) = se + sf$. Since $s, se \in S$ and $sf \in M$, we have $s = se$. Similarly, $s = es$, so e is a multiplicative identity element of S. Since the multiplicative identity element of S is unique, the remaining statements in the Proposition follow immediately. In particular, $e^2 = e$ and so $ef = e(1 - e) = e - e^2 = 0$. If $\mathcal{E} \colon R \to R$ is an S-bimodule map as in Proposition 2.1.2, then for any $s \in S$, $s\mathcal{E}(1) = \mathcal{E}(s1) = \mathcal{E}(s) = s$. Similarly, $\mathcal{E}(1)s = \mathcal{E}(1s) = \mathcal{E}(s) = s$, so $\mathcal{E}(1)$ is also a multiplicative identity element of S. It follows that $\mathcal{E}(1) = e$ because the multiplicative identity element of a ring with identity is unique.

We will use the following lemma to prove Proposition 2.5.2.

Lemma 2.1.4. Let R be a ring with unit 1. If $S \subseteq R$ is a corner with a complement M and $1 = e + f$ for $e \in S$, $f \in M$, then $fR + Rf \subseteq M$.

Proof. For $x \in R$ we have $\mathcal{E}(fx) = e\mathcal{E}(fx) = \mathcal{E}(efx) = \mathcal{E}(0) = 0$ since e is the identity for S. Thus $fR \subseteq M$. Similarly, $\mathcal{E}(xf) = \mathcal{E}(xf)e = \mathcal{E}(xfe) = \mathcal{E}(0) = 0$, and so $Rf \subseteq M$.

There are four special classes of corners which are of particular importance to the theory. Those are *Peirce corners*, *unital corners*, *rigid* and *split corners*, and they are defined as follows.

Definition 2.1.5 (Lam [Lam, 2006])**.** Let R be a ring, $S \subseteq R$ a corner of R.

1. S is called a *Peirce corner* if there exists $e \in R$ such that $e^2 = e$ and $S = eRe$.

2. When R has an identity element, then (so does S by Proposition 2.1.3 and) S is called a *unital corner* if the identity element of R belongs to S.

3. S is called a *rigid corner* if there exists a unique complement of S in R (two complements M and M' are considered identical if $M = M'$).

4. S is called a *split corner* if there exists a complement of S in R which is an ideal in R (equivalently, if there exists a complement of S in R which is a subring of R; in this case we have a unital ring isomorphism $S \cong R/M$).

Furthermore, if R is a ring and $e_1, e_2, \ldots, e_n \in R$ are idempotents with $e_i e_j = 0$ for $i \neq j$, then $\bigoplus_{i=1}^{n} e_i R e_i$ is a corner of R, which will be called a *generalised Peirce corner*.

We note the following fact.

Fact 2.1.6 (Unitalization)**.** If R is a non-unital ring, its unitalization is the abelian group $R^\sharp = R \oplus \mathbb{Z}$. With the multiplication $(x, m)(y, n) = (xy + nx + my, mn)$ $(x, y \in R;\ m, n \in \mathbb{Z})$, $R^\sharp$ is a unital ring with unit $(0, 1)$, and $R \subseteq R^\sharp$ is an ideal. If $S \subseteq R$ is a corner and $\mathcal{E}\colon R \to R$ is a corresponding Lam conditional expectation with $\mathcal{E}(R) = S$ (see Proposition 2.1.2), then

$$\mathcal{E}^\sharp : R^\sharp \to R^\sharp \text{ defined by } \mathcal{E}^\sharp(x, m) = (\mathcal{E}(x), m)\ (x \in R, m \in \mathbb{Z})$$

is a Lam conditional expectation from $R^\sharp$ onto $\mathcal{E}^\sharp(R^\sharp) = S^\sharp = S \oplus \mathbb{Z}$. In other words, if S is a corner of R then $S^\sharp$ is a corner of $R^\sharp$. Furthermore, if $\mathcal{E}$ is multiplicative then $\mathcal{E}^\sharp$ is also multiplicative. (Compare this fact with Proposition 3.3.1.)

In light of the following theorem, the theory of general corners may be reduced to Peirce corners and unital corners in many ways.

Theorem 2.1.7 (Lam [Lam, 2006])**.** If R is a unital ring, then:

1. Any corner of R is a unital corner of some Peirce corner of R; and
2. Any corner of R is a Peirce corner of some unital corner of R.

For a detailed proof of these results we refer the reader to [Lam, 2006, Section 5]. Here we briefly outline the key steps. As regards the first result, if S is a corner of a unital ring R, then $R = S \oplus M$ for some group $M \subseteq R$. Therefore the identity 1 of R can be uniquely written as $1 = e + f$ for $e \in S$, $f \in M$. By Proposition 2.1.3, the summand e is an identity element of S and eRe. Since $S \subseteq eRe$, it follows that S is a unital corner of the Peirce corner eRe of R.

As regards the second result, that any corner of R is a Peirce corner of some unital corner of R, assume again that S is a corner of a unital ring R. Then $R = S \oplus M$ for some group $M \subseteq R$. Since the identity 1 of R can be uniquely written as $1 = e + f$ for $e \in S$, $f \in M$ with $ef = fe = 0$ (Proposition 2.1.3), it follows that $S \cap fRf = 0$. Indeed, if $x \in S \cap fRf$ then $x = ex = f(ex)$ because e is an identity of S and f is an identity of fRf. Thus $x = 0$ because R is an associative ring and $fe = 0$. Now, $S' = S \oplus fRf$ is a unital corner of R and $S = eS'e$ is a Peirce corner of S'.

We shall close this section by noting that H. Kraft, L. W. Small, and N. R. Wallach have proved in [Kraft et al., 1999] a number of important results regarding corner rings (however in [Kraft et al., 1999] the the term 'corner rings' has not been explicitly used). In particular, the authors have showed that when S is a subring of a ring R with $R = S \oplus V$, and V has certain invariance properties, then some properties of R are inherited by S. For instance, if S is a subring of a semisimple ring R, and if S is also a bimodule direct summand of R then S is also semisimple i.e., a corner of a semisimple ring is semisimple (we will use this in Corollary 3.9.2 and in Lemma 3.12.2).

We shall also point out that there is a different definition and approach to 'corners rings' which was proposed by S. A. Amitsur and L. W. Small in [Amitsur and Small, 1990]. Here is the definition:

Definition 2.1.8. Let R be a ring. A subring S of R is called an *Amitsur-Small corner* of R if $SRS \subseteq S$.

Amitsur-Small corners also generalize subrings of the form eRe to the idea of 'corners'. In other words, every Peirce corner eRe is an Amitsur-Small corner. But there exist corners that are not Amitsur-Small corners and vice versa. For example, the algebra of finite-rank operators $\mathcal{F}(H)$ on an infinite-dimensional Hilbert space H is an Amitsur-Small corner in $\mathcal{B}(H)$ – the algebra of all bounded linear operators on H. Also its norm-closure, the algebra of compact operators $\mathcal{K}(H)$, is an Amitsur-Small corner in $\mathcal{B}(H)$. But neither $\mathcal{F}(H)$ nor $\mathcal{K}(H)$ are corners of $\mathcal{B}(H)$ in the sense of Definition 2.1.1 because they are not unital. On the other hand, if R is any ring with a unit-element 1 and S is a unital proper corner of R, then $SRS \nsubseteq S$ and so S is not an Amitsur-Small corner.

We also note that an unital corner need not be a Peirce corner in general. For example, let $C([-1,1])$ denotes the C^*-algebra of continuous complex-valued functions on the interval $[-1,1]$. Since $C([-1,1]) = C_{\text{even}}([-1,1]) \oplus C_{\text{odd}}([-1,1])$, the direct sum of even and odd functions, and the product of an even function and an odd function is an odd function, it follows that $C_{\text{even}}([-1,1])$ is a unital corner of $C([-1,1])$ (in the sense of Definition 2.1.1). But $C_{\text{even}}([-1,1])$ is not a Peirce corner of $C([-1,1])$ because there are no nontrivial idempotents in $C([-1,1])$ (the only idempotents of $C([-1,1])$ are the '0' and '1' functions). (Note that $C_{\text{even}}([-1,1])$ is not an Amitsur-Small corner of $C([-1,1])$.) Note also that the space $L^2(\mathbb{R})$ of square-integrable functions with respect to the Lebesgue measure on the real line $\mathbb{R}$ and with the inner product $\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$, for f and g in $L^2(\mathbb{R})$, is the orthogonal direct sum of the space S of even functions and the space M of odd functions. The orthogonal projection $\mathcal{E}$ of $L^2(\mathbb{R})$ onto the corner S along M is given by $\mathcal{E}(f)(x) = (f(x)+f(-x))/2$ for $f \in L^2(\mathbb{R})$, $x \in \mathbb{R}$. (We could also consider the orthogonal projection $\mathcal{E}_h(f)(x) = \overline{h(x)}[h(x)f(x) \pm h(-x)f(-x)]$ where $|h(x)|^2 + |h(-x)|^2 = 1$.)

2.2 Uniqueness of Complements

In this section we present new proofs of T. Y. Lam's results [Lam, 2006, (2.8)(2), (5.11)] on uniqueness of complements in arbitrary (not necessarily unital) rings.

Theorem 2.2.1. A Peirce corner $S = eRe$ in a ring R (where $e \in R$ is an idempotent) has a unique complement $M = R(1-e)+(1-e)R = eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$.

Proof. Let M be any complement for S. Then, for $x \in R$ we have $x = s + m$ with $s \in S$, $m \in M$. So $exe = ese + eme =$

$s + eme$. But $eme \in M$ since $e \in S$ and so $eme = exe - s \in S \cap M = \{0\}$. Hence $s = exe$ and $m = x - exe = ex(1-e) + (1-e)x = ex(1-e) + (1-e)xe + (1-e)x(1-e)$.

The summands $eR(1-e)$, $(1-e)Re$ and $(1-e)R(1-e)$ are easily seen to each have trivial intersection with the sum of the other two.

Theorem 2.2.2. If R is a ring and $e_1, e_2, \ldots, e_n \in R$ are idempotents with $e_i e_j = 0$ for $i \neq j$, then the generalised Peirce corner $S = \bigoplus_{i=1}^{n} e_i R e_i$ has a unique complement.

The unique Lam conditional expectation $\mathcal{E}\colon R \to R$ with range S is given by $\mathcal{E}(x) = \sum_{i=1}^{n} e_i x e_i$.

Proof. Let M be a complement for S with corresponding Lam conditional expectation $\mathcal{E}_0$. The idempotent $e = \sum_{i=1}^{n} e_i$ is the identity element for S and it follows from $R \ni x = s + m$ ($s \in S$, $m \in M$) that $exe = s + eme$ with $eme \in M$. So $\mathcal{E}_0(x) = s = \mathcal{E}_0(exe)$.

Note that for $z \in S$ we have $z = \sum_{k=1}^{n} e_k z e_k$.

For $y \in eRe$ we have $y = eye = \sum_{i,j=1}^{n} e_i y e_j = \sum_{i=1}^{n} e_i y e_i + \sum_{i \neq j} e_i y e_j$. For $i \neq j$ we have $\mathcal{E}_0(e_i y e_j) = \sum_{k=1}^{n} e_k \mathcal{E}_0(e_i y e_j) e_k = \sum_{k=1}^{n} \mathcal{E}_0(e_k e_i x e_j e_k) = 0$. Hence $e_i y e_j \in M$ for $i \neq j$ and $\mathcal{E}_0(y) = \sum_{i=1}^{n} e_i y e_i$.

It follows that for $x \in R$, $\mathcal{E}_0(x) = \mathcal{E}_0(exe) = \sum_{i=1}^{n} e_i exe e_i = \sum_{i=1}^{n} e_i x e_i$.

2.3 Annihilators and Split Corners

For a nonempty subset S of a ring R we define the *annihilator* of S as follows $S^\perp := \{x \in R : xs = 0 = sx \text{ for all } s \in S\}$. Note that $S^\perp$ is a subgroup of the additive group of R, and if S is a two-sided ideal of R then so is $S^\perp$.

Proposition 2.3.1. Let R be a ring and S a two-sided ideal of R such that $S \cap S^\perp = \{0\}$. Then S is a corner of R if and only if $R = S \oplus S^\perp$.

Proof. Suppose S is a corner of R. Then $R = S \oplus M$ for some subgroup $M \subseteq R$ with $SM \subseteq M$, $MS \subseteq M$. Since S is a two-sided ideal of R we also have that $SM \subseteq S$, $MS \subseteq S$. Therefore $SM \subseteq S \cap M = \{0\}$ and $MS \subseteq S \cap M = \{0\}$, so $R = S + S^\perp$ because $M \subseteq S^\perp$. Finally, the assumption $S \cap S^\perp = \{0\}$ yields that $R = S \oplus S^\perp$.

For the converse, if $R = S \oplus S^\perp$, then S is a corner of R since $SS^\perp = S^\perp S = \{0\} \subseteq S^\perp$.

We note that if a corner S in R is complemented by a subring, say M, of R, then M is automatically a two-sided ideal of R. Indeed, suppose $R = S \oplus M$ with S a corner of R and the complement M a subring of R. Then $RM = (S \oplus M)M \subseteq SM + MM \subseteq M + M = M$, so M is a left ideal of R. We also have that $MR = M(S \oplus M) \subseteq MS + MM \subseteq M + M = M$, so M is a right ideal of R. Hence M is a two-sided ideal of R.

Corollary 2.3.2. Assume that S is a corner of a ring R. If S has a complement M that is a ring, then S is a split corner and $R/M \cong S$.

2.4 Graphs of Ring Homomorphisms

The new idea that we present in this section, and which we will develop further in future sections (§5.5, §4.3), is to consider graphs of ring homomorphisms $\phi\colon R_1 \to R_2$ as corner subrings of $R_1 \oplus R_2$ (here R_1 and R_2 are rings). We will show that one can always construct a Lam conditional expectation $\mathcal{E}\colon R_1 \oplus R_2 \to R_1 \oplus R_2$ onto the graph of ϕ and that this $\mathcal{E}$ must be multiplicative. In a similar way, we show that one can also construct a Lam conditional expectation $\mathcal{E}\colon R \oplus M \to R \oplus M$ onto the graph of a derivation $\delta\colon R \to M$ (here R is a ring and M is an R-bimodule). Similar constructions are valid in the context of Lie rings.

Suppose we have a ring homomorphism $\phi\colon R_1 \to R_2$ where R_1 and R_2 are rings.

Let $R = R_1 \oplus R_2$, a ring with the summands as ideals. The subring $R_1 \times \{0\}$ of $R_1 \oplus R_2$ is isomorphic to R_1 and so is identified with R_1; similarly for $\{0\} \times R_2$.

Denote by S the graph of ϕ, that is, $S = \{(x, \phi(x)) : x \in R_1\}$, and define $\mathcal{E}\colon R \to R$ by $\mathcal{E}(a, b) = (a, \phi(a))$ $(a \in R_1, b \in R_2)$. Note the following:

1. S is a subring of R. Indeed, if $x, y \in R_1$ then $(x, \phi(x)) + (y, \phi(y)) = (x+y, \phi(x)+\phi(y)) = (x+y, \phi(x+y)) \in S$, and $(x, \phi(x))(y, \phi(y)) = (xy, \phi(x)\phi(y)) = (xy, \phi(xy)) \in S$.

2. $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. Indeed, if $a \in R_1$ and $b \in R_2$ then $\mathcal{E} \circ \mathcal{E}(a, b) = \mathcal{E}(a, \phi(a)) = (a, \phi(a)) = \mathcal{E}(a, b)$.

3. $\mathcal{E}$ is an S-bimodule map. Indeed, if $x, a \in R_1$ and $b \in R_2$ then $\mathcal{E}((x, \phi(x))(a, b)) = \mathcal{E}(xa, \phi(x)b) = (xa, \phi(xa)) = (xa, \phi(x)\phi(a)) = (x, \phi(x))(a, \phi(a)) = (x, \phi(x))\mathcal{E}(a, b)$. Similarly, $\mathcal{E}((a, b)(x, \phi(x))) = \mathcal{E}(a, b)(x, \phi(x))$.

4. $\mathcal{E}(R) = S$ (a consequence of the definition of $\mathcal{E}$).

5. $\mathcal{E}$ is multiplicative. Indeed, if $a, c \in R_1$ and $b, d \in R_2$ then $\mathcal{E}((a, b)(c, d)) = \mathcal{E}(ac, bd) = (ac, \phi(ac)) = (ac, \phi(a)\phi(c)) = (a, \phi(a))(c, \phi(c)) = \mathcal{E}(a, b)\mathcal{E}(c, d)$.

6. If R_1 and R_2 are rings with involution, then $R = R_1 \oplus R_2$ becomes a ring with involution $(a, b) \mapsto (a^*, b^*)$ $(a \in R_1, b \in R_2)$. Moreover, if ϕ is *-preserving then $\mathcal{E}$ is *-preserving and S is self-adjoint. Indeed, if $a \in R_1$ and $b \in R_2$ then $\mathcal{E}((a, b)^*) = \mathcal{E}(a^*, b^*) = (a^*, \phi(a^*)) = (a^*, \phi(a)^*) = (a, \phi(a))^* = (\mathcal{E}(a, b))^*$. If $x \in S$ then $(x, \phi(x)) \in S$ and ϕ is *-preserving then $(x, \phi(x))^* = (x^*, \phi(x)^*) = (x^*, \phi(x^*)) \in S$, so S is self-adjoint.

We have shown:

Proposition 2.4.1 (Lam conditional expectations onto graphs of ring homomorphisms)**.** If $\phi\colon R_1 \to R_2$ is a ring homomorphism, then the graph of ϕ, $S = \{(x, \phi(x)) : x \in R_1\}$, is a corner of $R := R_1 \oplus R_2$.

If R_1 and R_2 are rings with involutions and ϕ is involution-preserving, then the corner S is self-adjoint.

Proposition 2.4.1 asserts in particular that one can always find a Lam conditional expectation onto the graph of a ring homomorphism (between two rings); moreover, this expectation is multiplicative. In fact, we will show (Proposition 2.4.2) that a Lam conditional expectation is multiplicative if and only if its kernel is an ideal.

Proposition 2.4.2. Let R be a ring, $\mathcal{E}\colon R \to R$ an additive map with $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$ and

$$\mathcal{E}(\mathcal{E}(x)y) = \mathcal{E}(x)\mathcal{E}(y), \qquad \mathcal{E}(x\mathcal{E}(y)) = \mathcal{E}(x)\mathcal{E}(y)$$

for $x, y \in R$. Then $\mathcal{E}$ is multiplicative if and only if $\ker \mathcal{E}$ is an ideal.

Proof. If $\ker \mathcal{E}$ is an ideal in R, then for $x, y \in R$ we have $\mathcal{E}(m) = 0$, where $m = x - \mathcal{E}(x)$, and $\mathcal{E}(xy) = \mathcal{E}((m + \mathcal{E}(x))y) = \mathcal{E}(my + \mathcal{E}(x)y) = \mathcal{E}(my) + \mathcal{E}(\mathcal{E}(x)y) = \mathcal{E}(x)\mathcal{E}(y)$.

Conversely, if $\mathcal{E}$ is multiplicative, then for $x \in R$, $m \in \ker \mathcal{E}$, we have $\mathcal{E}(xm) = \mathcal{E}(mx) = 0$ thus $xm, mx \in \ker \mathcal{E}$.

Question. Is every corner the graph of a ring homomorphism if the corresponding Lam conditional expectation $\mathcal{E}$ is multiplicative?

Proposition 2.4.3 (Lam conditional expectations onto graphs of derivations)**.** Let R be a ring and let M be an R-bimodule. Then $R \oplus M$ is a ring with the operations: $(x, m) + (y, n) = (x + y, m + n)$ and $(x, m) \cdot (y, n) = (xy, xn + my)$ for $x, y \in R$, $m, n \in M$. Let $\delta \colon R \to M$ be a derivation (that is: a function satisfying $\delta(x + y) = \delta(x) + \delta(y)$ and $\delta(xy) = \delta(x)y + x\delta(y)$ for $x, y \in R$). Then the graph of δ, i.e. the set $S = \{(x, \delta(x)) : x \in R\}$, is a (corner) subring of $R \oplus M$, and the map $\mathcal{E} \colon R \oplus M \to R \oplus M$ defined by $\mathcal{E}(x, m) = (x, \delta(x))$ for $x \in R$, $m \in M$, is a Lam conditional expectation onto S. Moreover, $\mathcal{E}$ is a ring homomorphism.

Here is another model for corners and corresponding 'expectations' in the context of Lie rings: If R is a ring, write $[x, y] = xy - yx$ for the Lie product of two elements $x, y \in R$ (or consider a 'general' Lie bracket on R), and consider $R = (R, [\cdot, \cdot])$ as a Lie ring. Then a Lie subring, i.e. a subgroup $L \subseteq R$ of the additive group of R with $[L, L] \subseteq L$, may be called a *Lie corner* if $R = L \oplus M$ and $[L, M] \subseteq M$ for some subgroup $M \subseteq R$. Note that L is a Lie corner if and only if there exists an additive group homomorphism $\mathcal{E} \colon R \to R$ such that $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$ and $\mathcal{E}([l, x]) = [l, \mathcal{E}(x)]$, $\mathcal{E}([x, l]) = [\mathcal{E}(x), l]$ for $l \in L$, $x \in R$; such $\mathcal{E}$ may be called a *Lie conditional expectation.*

An example of a Lie corner is the space Skew_n of n-by-n

skew-symmetric matrices ($A^T = -A$). Indeed, Skew_n is a Lie subring of Mat_n, the space of all n-by-n matrices, and $\mathrm{Mat}_n = \mathrm{Skew}_n \oplus \mathrm{Sym}_n$ (since $A = \frac{1}{2}(A-A^T)+\frac{1}{2}(A+A^T)$), where Sym_n denotes the space of symmetric matrices ($A^T = A$). The corresponding Lie conditional expectation is the map $\mathcal{E}\colon \mathrm{Mat}_n \to \mathrm{Mat}_n$ given by $\mathcal{E}(A) = \frac{1}{2}(A - A^T)$.

Proposition 2.4.4 (Lie conditional expectations onto graphs of Lie homomorphisms)**.** Let R_1, R_2 be rings considered as Lie rings, and let $\phi\colon R_1 \to R_2$ be a Lie homomorphism (i.e. an additive group homomorphism with $\phi([x, y]) = [\phi(x), \phi(y)]$ for $x, y \in R_1$). Consider $R = R_1 \oplus R_2$, a Lie ring with $[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$ for $x_1, y_1 \in R_1$, $x_2, y_2 \in R_2$. Then the graph of ϕ, i.e the set $L = \{(x, \phi(x)) : x \in R_1\}$, is a Lie corner of R, and the map $\mathcal{E}\colon R \to R$ defined by $\mathcal{E}(x_1, x_2) = (x_1, \phi(x_1))$ ($x_1 \in R_1,\ x_2 \in R_2$) is a Lie conditional expectation onto L. Moreover $\mathcal{E}$ is a Lie homomorphism, i.e. $\mathcal{E}([(x_1, x_2), (y_1, y_2)]) = [\mathcal{E}(x_1, x_2), \mathcal{E}(y_1, y_2)]$ for $x_1, y_1 \in R_1$, $x_2, y_2 \in R_2$.

2.5 $\mathbb{Q}$ is a Dense Corner in $\mathbb{R}$

Consider $\mathbb{R}$, the set of real number, as a vector space over the rationals $\mathbb{Q}$. Let B be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ including 1. Let M be the $\mathbb{Q}$-linear span of $B \setminus \{1\}$. Then $\mathbb{Q}1 = \mathbb{Q}$ and

$$\mathbb{R} = \mathbb{Q} \oplus M.$$