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Part II. **Studying Neuroscience Models via the CNN Approach**

PREFACE

In the first part of this book mathematical modelling of some natural phenomena such as tsunami waves and tornado dynamics is presented. The study of the propagation of tsunami from their small disturbance at sea level to the size they reach approaching the coast has interested many scientists. It is clear that in order to predict accurately the appearance of tsunami it is fundamental to build up a good model. The thrust of the mathematical approach is to examine how a wave, once initiated, moves, evolves and eventually becomes such a destructive force of nature. The previous considerations show that from initiation to reaching the region of the shore, a good approximation of tsunami waves is provided by the solutions of the corresponding model equation. In the original physical variables this means that up to the near-shore the wave profile remained unaltered propagating at constant speed. The linear model breaks down when the tsunami waves enter the shallower water of the shore regions. Therefore, the appropriate equations are those modelling the propagation of long water over variable depth. In this region faster wave fronts can catch up slower ones and this can result in large amplitude wave fronts building up behind the smaller ones.

Even with the aid of the most advanced computers it is not possible to find the exact solutions to the non-linearity governing equations for water waves. For this purpose we introduce the Cellular Nonlinear Network (CNN) approach. Quantifying the dynamics of tsunami waves as they impact on shore areas is a challenging mathematical and physical problem of the outmost importance. In this part of the book we shall discuss the different mathematical models of tsunami waves, such as KdV, shallow water equations, Camassa-Holm (CH), long water waves with nonlinear vorticity, the two-component CH system, etc. In order to study the dynamics of our models we use the CNN approach to discretize the governing equation over a suitable grid.

In this part of the book we shall present one more model—tornado dynamics. Observations of the tornado have a rich history, provided by many papers only for the 20th century. Observations of the actual tornado indicate a strong non-linearity and non-equilibrium of processes in the atmosphere during the formation and existence of a tornado that do not allow the creation of the perfect model of this exotic phenomenon. To gain some insight into the processes involved we shall set up a numerical approach via the Cellular Nonlinear Network (CNN) that treats a vortex as a fluid dynamical system.

In the second part of the book we shall study some models arising in neuroscience. Nonlinear reaction-diffusion types of equations are widely used to describe phenomena in different fields, such as the biology-Fisher model, the Hodgkin-Huxley model and its simplification—the FitzHugh Nagumo nerve conduction model, etc. The famous Hodgkin-Huxley neuron model is the first mathematical model describing neural excitation transmission derived from the angle of physics and lays the basis of electrical neurophysiology. The FitzHugh Nagumo equation, which is a simplification of the Hodgkin-Huxley model, describes the generation and propagation of the nerve impulse along the giant axon of the squid. The FitzHugh Nagumo systems are of fundamental importance for understanding the qualitative nature of nerve impulse propagation. In this part of the book we shall study a coupled FitzHugh Nagumo neural system and the phenomenon "edge of chaos". It was shown that the difference in behaviour is due to different bifurcation mechanisms of excitability. For dynamical systems in neuroscience, the type of bifurcation determines the computational properties of neurons. Based on the finite propagating speed in the signal transmission between the neurons, in this part of the book we shall present various FitzHugh Nagumo neural systems and study their dynamics via CNN approach.

The book is addressed to a broader audience including graduate students, PhD students, mathematicians, physicists, engineers and specialists in the domain of Partial Differential Equations (PDE) and their applications. Certainly, there are monographs on nonlinear PDE based on complicated and difficult methods that make reader acceptance hard, especially for beginners and non-specialists on PDE. In this sense the proposed book aims to provide a simpler approach, based on Cellular Nonlinear Networks for modelling and studying nonlinear waves such as tsunami waves, tornados and travelling waves in FitzHugh Nagumo neural systems.

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PART I.

MODELLING ENVIRONMENTAL PROBLEMS VIA THE CELLULAR NONLINEAR NETWORKS APPROACH

CHAPTER ONE

STUDY OF SHALLOW WATER WAVES

Introduction

The December 2004 tsunami, generated on Sunday, 26 December 2004 at 7:58 am (local time, Indonesia) by the most powerful earthquake in decades, killed more than 275,000 people [124] and made millions homeless, making it one of the most destructive natural disasters in history. The hypocentre of the earthquake was about 30 km below the floor of the Indian Ocean, at 160 *km* off the west coast of the Indonesian island of Sumatra, and the violent movement of the Earth's tectonic plates displaced an enormous amount of water, sending tsunami waves westwards across the Indian Ocean as well as eastwards across the Andaman Basin. Since the earthquake occurred over about 10 minutes along a 1000 *km* long approximately straight fault line, the generated tsunami waves were approximately two-dimensional [106, 107], that is, the motion was approximately identical in any direction parallel to the crest line. Within hours these waves crashed upon the coastline of 11 Indian Ocean countries, snatching people out to sea, drowning others in their homes or on beaches, and demolishing property from South Africa to Thailand. The catastrophic devastation wrought by the tsunami occurred primarily on the shores of the Bay of Bengal and of the Andaman Basin but substantial damage was also documented in Somalia (some 5000 *km* to the west of the epicentre) and large waves were noticed as far as Madagascar and South Africa. For modelling purposes, outside of the Bay of Bengal the two-dimensional character of the tsunami waves can no longer be taken for granted since diffraction around islands and reflection from steep shores alter this feature considerably. The earthquake that generated the tsunami changed the shape of the ocean floor by raising it by a few metres to the west of the epicentre and lowering it to the east (over 100 *km* in the east-west direction and about 900 *km* in the north-south direction) [107]. The initial shape of the wave pattern that developed into the tsunami wave featured therefore to the west of the epicentre a wave of elevation followed by a wave of depression (that is, with water levels higher and respectively lower than normal), while to the east of the epicentre the initial wave profile consisted of a depression followed by an elevation. The tsunami waves to the west of the epicentre propagated approximately 1600 *km* across the Bay of Bengal in the Indian Ocean towards India and Sri Lanka, hitting in less than 3 hours the coastal regions of India and Sri Lanka, the first tsunami wave being a wave of elevation [37, 107]. The tsunami wave to the east of the epicentre crossed the 700 *km* distance across the Andaman Basin in less than 2 hours, with a leading wave of depression as it hit resorts in Thailand [41, 107]. Several reports of seaside villages in Thailand confirmed that the first evidence of the tsunami was that the ocean receded rapidly and far. Many people were killed because they went to view the retreating ocean exposing the seafloor, unaware that the large wave of depression would be followed by several large waves of elevation (photographs reproduced in [37] show that the shoreline receded before the arrival of the first wave front at Hat Ray Leh beach in southern Thailand and two fronts, one closely behind the other and the second considerably larger, occurred at this time with the maximum height of the tsunami, as it came ashore at this location, of about 10 *m*). The fact that, as the tsunami waves reached the shore in either direction, the shape of the initial disturbance (first a wave of elevation, then a wave of depression, respectively *vice-versa*) was not altered is of utmost importance in validating a theory for the wave dynamics on this occasion. This observation suggests that perhaps the shape of the tsunami waves remained approximately constant as they propagated across the Bay of Bengal and across the Andaman Basin. This hypothesis is further substantiated by the accurate measurements of the water's surface (the spatial scale of the coverage being about 800 *km*) performed by a radar altimeter on board a satellite about two hours after the earthquake took place, along a track traversing the Indian Ocean/Bay of Bengal [41]. These clearly show a leading wave of elevation, followed by a wave of depression, a feature common both to the initial wave profile west of the epicentre and to the tsunami as it entered the coastal regions of India and Sri Lanka. These measurements also confirm another essential feature of tsunami waves: even though these waves reach large amplitudes due to the diminishing depth effect as they approach the shore (waves as high as 30 *m* were observed near the city Banda Aceh on the west coast of the northern tip of Sumatra [70] about 160 km away from the epicentre of the earthquake), tsunami waves are barely noticeable at sea due to their small amplitude. Indeed, the satellite data show that the maximum amplitude of the waves, whether positive or negative with respect to the usual sea level, was less than 0.8*m* over distances of more than 100 *km*. To get a sense of how mild this disturbance is, we point out the delightful argument from [107]: sitting in a boat in the Bay of Bengal, midway between Sumatra and Sri Lanka, it would take a tsunami wave component (whether a wave of elevation/depression) with a wavelength of 100 *km* about 10 minutes to move past the boat, time in which the boat would move up/down by 0.8*m*, and then back down/up by 0.8*m*. For these estimates the assumption was made (see [107]) that the tsunami wave speed is at least 620 *km/h*, a hypothesis that is confirmed by the considerations made in the next chapters.

To predict accurately the appearance of a tsunami it is of paramount importance to model these powerful waves, explaining the propagation mechanism as well as the process by which they evolve from a small-amplitude disturbance of the sea level (albeit one of large wavelengths, in excess of 100 *km*) to become such devastating forces of nature as they approach the coast.

Shallow water theory

Perhaps the most important scientific discovery of the last decades in the context of water waves was soliton theory. Solitons arise as special solutions of a widespread class of weakly nonlinear dispersive partial differential equations modelling water waves, such as the Korteweg-de Vries (KdV) [48] or Camassa-Holm (CH) equation [18], representing to various degrees of accuracy approximations to the governing equations for water waves in the shallow water regime (see the discussion in [38]). Informally, dispersion means that different harmonic components of a solution travel at different velocities determined by the frequency, so that within the framework of linear theory, even though energy is preserved due to the neglecting of dissipative effects, the different components of a solution spread out and consequently the solution at later times tends to have a much smaller amplitude than initially. At the weakly nonlinear level, however, in certain regimes, non-linearity balances dispersion and permanent and localized wave forms travelling at constant speed ("solitary waves") arise as solutions. If such solitary waves present elastic interaction in the sense that as a result of the nonlinear interaction with other waves of this type, they emerge from the collision unchanged, except for a phase shift, we say that the solitary waves are solitons [48]. Examples of equations relevant to water waves with soliton solutions are KdV and CH. When first encountered, the situation we refer to might seem perplexing: in talking about the propagation of shallow water waves, in addition to KdV there exists the regularized long wave equation [82] (usually called

BBM [11]), which provides an approximation of the governing equations for water wave equations in the shallow water regime of the same accuracy as KdV (CH arises as a higher-order approximation [38]), whose solitary waves have even similar expressions. The solitary waves of KdV equation are solitons [48] while those of BBM are not [77]. This apparent paradox is however easily resolved. Given a physical situation, under certain simplifying assumptions established laws from physics can be applied to obtain a model of the physical process. Investigating the behaviour of the model by mathematical methods, our understanding of the physical phenomenon can be improved. The conclusions reached will reflect reality (that is, specific physical situations which may be observed experimentally) only insofar as the accuracy of the model permits: the value of a model depends on the number of physically useful deductions which can be made from it. The "truth" of the model is meaningless as all experiments contain inaccuracies of measurement and effects other than those accounted for cannot be totally excluded. Even with the aid of the most advanced computers it is not possible to find exact solitary wave solutions to the nonlinear governing equations for water waves. The progress towards understanding solitary waves based on the governing equations for water waves is noticeable: localized disturbances of a flat water surface propagating without a change of form have to be two-dimensional [42], the existence of two-dimensional solitary wave solutions was established [2], these waves have to be waves of elevation with a profile symmetric about the crest [30], and a qualitative description of the particle motion beneath the solitary wave is available [35]. Since an in-depth study of solitary wave interactions using the governing equations is not within reach, to shed light on this important aspect one has to perform approximations leading to simplified model equations.

The linear theory of waves of small amplitude does not provide any approximation to solitary waves (see [114]), so nonlinear approximations to the governing equations for water waves have to be made: KdV and BBM arise as weakly nonlinear approximations, with CH capturing more nonlinear effects [38]. In assessing the relative importance of these model equations the benchmark is provided by the degree to which they provide a description/explanation of specific important water wave phenomena encountered in nature. Soliton interactions of water waves occur in nature and can be reproduced in the laboratories, with the predictions made by KdV and BBM in close agreement with experimental measurements [106]. The solitary waves of KdV are orbital stable (meaning that their shape is stable under small perturbations) [9], [72], a feature valid also for BBM [9] and CH [40], [50], which explains why these wave patterns are physically

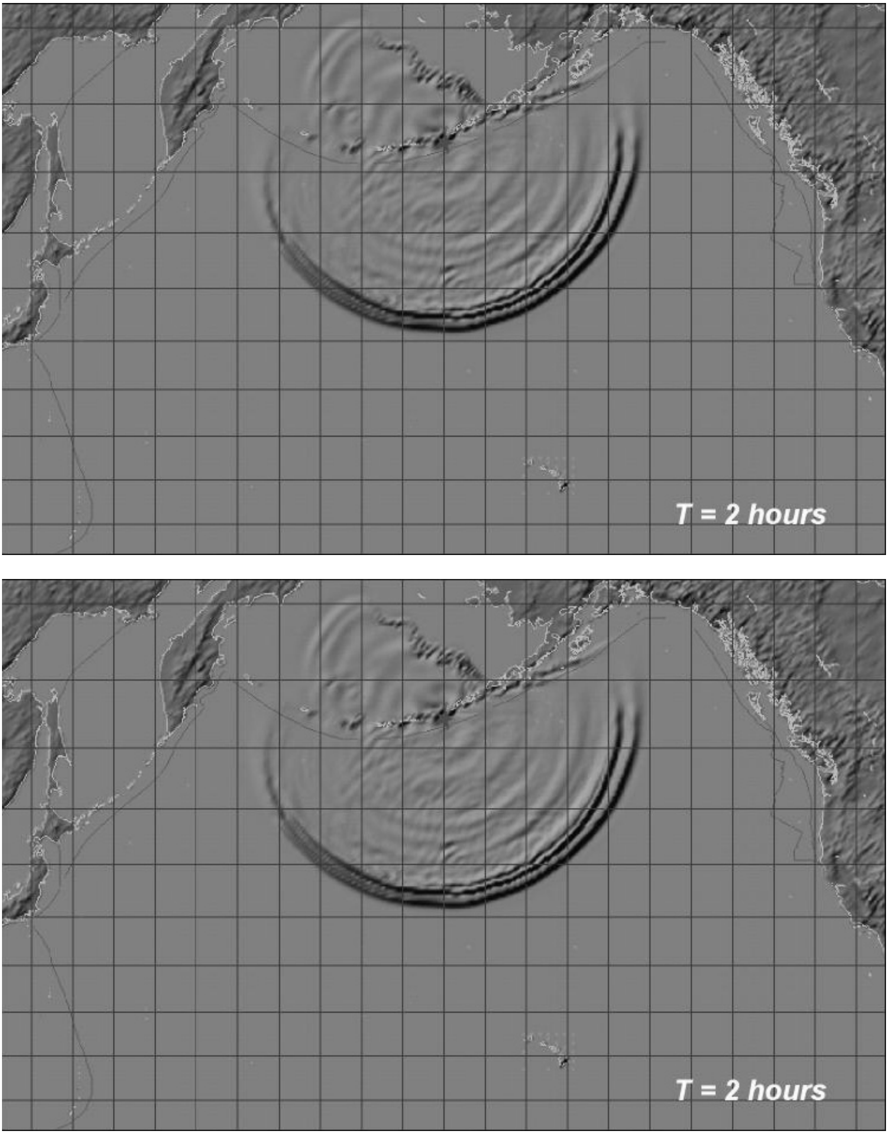

Figure 1.1 A tsunami as a large soliton

recognizable. The importance of KdV is further enhanced by the fact that an inverse scattering analysis which relies on structural properties of the equation (e.g. its Hamiltonian structure and associated integrals of motion) leads to the following dynamical picture: starting with arbitrary initial data that are smooth and sufficiently localized in space, the KdV solution that evolves from these data is developing into a finite number of localized solitary waves (solitons), plus an oscillatory tail (see [48]). Each solitary wave retains its localized identity and the taller waves travel faster than the smaller ones, while the oscillatory tail disperses and spreads out in space. Therefore the solution evolves into an ordered set of solitons, with the tallest in front, followed by an oscillatory tail that gradually fades out. This shows that the solitons are the key to understanding the dynamics of water waves as modelled by KdV. BBM is not integrable [80] so that the mechanism of solitary wave interactions is not as plain as for KdV. It is thus no accident that KdV plays a more important role in water wave theory than BBM (which remains a valid model equation but of more limited interest). As for CH, while it is integrable [34, 39], the dynamics of its soliton interactions is more intricate than for KdV so that it is mostly in the context of breaking waves that CH gained importance [13, 14, 34]. We are not concerned with this aspect here: we concentrate on the propagation of tsunami waves across the sea, and in this regard KdV is the proper model equation among the variety of shallow water models, as pointed out above.

We are thus led to the fundamental question of whether tsunami waves enter the regime of validity of KdV as an approximation to the governing equations for water waves. A frequent view encountered throughout the research literature could be formulated as

> ... a tsunami is produced by a large enough soliton. There may exist tsunamis not directly related to solitons but experts agree that the majority of registered tsunamis were produced by solitons. [51]

However, it is not just because KdV is a model arising in the shallow water regime for waves of small amplitude that one can regard tsunamis at sea as manifestations of solitons, even if this is implied by several classical as well as more recent research papers [41, 67, 105, and 116]. The question is whether the geophysical scales involved lead to time- and space-scales that are compatible with those required for KdV theory.

Shallow water classification

Let h be an average depth of the water, λ is a typical wavelength of the wave and a is typical amplitude. There are two important parameters $\varepsilon = \frac{a}{h}$, called the amplitude parameter, and the shallowness parameter $\delta = \frac{h}{\lambda}$. According to these parameters rigorous validity ranges can be obtained for the main physical regimes encountered in modelling the two-dimensional water waves:

1. Shallow-water, large amplitude ($\delta \ll 1$), ($\varepsilon \sim 1$), leading at first order to the shallow-water equations and at second order to the Green-Naghdi model. In this case when the wavelength $\delta \to 0$ is increasing the stability properties of travelling water waves improve significantly. Moreover, one can prove the orbital stability of these waves which allows their shape to be stable under small perturbations.

2. Shallow-water, medium amplitude regime ($\delta \ll 1$), ($\varepsilon \sim \delta$) leading to the Serre equations and to the Camassa-Holm equation (CH). Unlike KdV, which is derived by asymptotic expansions in the equation of motion, CH is obtained by using asymptotic expansions directly in the Hamiltonian for Euler's equations in the shallow water regime. The novelty of Camassa and Holm's work was the physical derivation of the CH equation and the discovery that the equation has solitary waves (solitons) that retain their individuality under interaction and eventually emerge with their original shapes and speeds. For this reason CH is not appropriate for advancing insight into the propagation of tsunamis.

3. Shallow-water, small amplitude or long-wave regime ($\delta \ll 1$), ($\varepsilon \sim \delta^2$) leading to the linear wave equation

$$\varphi_{tt} - \varphi_{xx} = 0 \qquad (1.1)$$

with the general solution

$$\varphi(x,t) = \varphi_+(x-t) + \varphi_-(x+t), \qquad (1.2)$$

where a sign $\pm$ refers to a wave profile $\varphi_\pm$ moving with unchanged shape to the right/left at constant unit speed. The small effects that were ignored at first order (small amplitude, long wave) build up on longer time/spatial scales to have a significant cumulative nonlinear effect so that on a longer time scale each of the waves that make up the solution (1.2) to (1.1) satisfies the KdV equation. This regime leads to Boussinesq systems as well.

4. Deep-water, small steepness regime ($\delta \gg 1$), ($\varepsilon\delta \ll 1$) leading to the full-dispersion Matsuno equations.

We are interested in the small-amplitude long waves, in the limits $\varepsilon \to 0$ and $\delta \to 0$. The regime $\varepsilon = O(\delta^2)$ emerges naturally since one obtains a problem involving only a small parameter ε. It is in this regime that at first order the evolution of the waves is governed by the linear wave equation (1.1) with the general solution (1.2). The corresponding dimensional speed is $\sqrt{gh_0}$. Let us choose the wave moving to the right, then by means of the method of multiple scales it is possible to obtain in the region of (x, t) space more precise information about the evolution of the water's free surface by taking into account weakly nonlinear interactions.

This can be achieved by showing that, to the next order of approximation the evolution of the leading order of the free surface is described by the KdV equation instead of the linear equation (1.1). The Boussinesq system is obtained by allowing the waves to travel in both directions. For our purposes it is more important to specify how solutions of KdV or Boussinesq approximate the free surface. The sharpest rigorous result in this direction is given in [66] and ensures that: given $\varepsilon > 0$, there exists $T_0 > 0$ such that if $\varepsilon = O(\delta^2)$, then if one defines

$$\varphi^{\varepsilon}(x,t) = \varphi^{+}(\tau, x - t),$$

where $\tau = t\,\varepsilon$ and $\varphi^{+}(\tau, \varphi)$ solves the KdV equation

$$\varphi^{+}_{\varphi} + \frac{3}{2}\,\varphi^{+}_{\varphi\varphi\varphi} + \frac{1}{6}\varphi^{+}\varphi^{+}_{\varphi} = 0$$

then for some $A > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ the following is satisfied:

$$|\varphi(x,t) - \varphi^{\varepsilon}(x,t)| \leq A\varepsilon^2 t, t \in \left[0, \frac{T_0}{\varepsilon}\right].$$

A similar approximation of order $O(\varepsilon^2 t)$ can be found for the solution of the Boussinesq system in the case when the wave propagation is not unidirectional. Moreover, in the case of a non-flat bed with small variations of the order of the size of the surface waves, meaning that if b measures the amplitude of the variations of the bottom topography, then

$\frac{b}{h_0} = O(\varepsilon)$, the constant-coefficient KdV equation may be replaced by a variable-coefficient KdV equation; and similarly for the Boussinesq system with the same scaling and approximation properties.

CHAPTER TWO

TSUNAMI MODELLING

Introduction

Tsunamis are without a doubt among the most infamous and least understood natural disasters today. Often referred to in the popular literature by the misnomer tidal wave, tsunamis are generated by large displacements in the sea level, often via seismic activity. Most tsunamis—a term from the Japanese for harbour wave—are caused by vertical movement along a break in the earth's crust (Fig. 2.1). Other causes can include volcanic collapse and subsidence, as well as landslides. Contrary to popular imagination, a tsunami needs to be neither large nor destructive—classification is based on the origin of the wave or wave period rather than on the size. Though there were more than 15,000 earthquakes recorded between 1861 and 1948, there were only 124 tsunamis. Indeed, off the west coast of South America, 1,098 earthquakes have led to only 20 recorded tsunamis. As waves of such great scale, generated by complex movements of the earth, and with such devastating consequences for populations surrounding the world oceans, the accurate modelling of tsunamis is of utmost importance.

One question which has been raised repeatedly is whether the behaviour of a tsunami at sea can be described by the Korteweg-de Vries equation. We will pursue this question for one of the greatest tsunamis of recorded history—generated by a series of earthquakes in southern Chile on May 22, 1960—as it propagated from Chile to Hawaii. These earthquakes, among them the largest ever recorded, resulted from a rupture about 1000 *km* long and 150 *km* wide along the fault between the Nazca and South American plates, at a focal depth of 33 *km*. The principal shock occurring on May 22 at 19:11 GCT registered at 9.5 on the moment magnitude scale, and led to changes in land elevation ranging from 6 *m* of uplift to 2 *m* of subsidence—which has been modelled to correspond to an average dislocation of 20 *m* along the fault, with peaks of more than 30 *m*. This subsidence extended as far as 29 *km* inland, resulting in some 10 km^2

of forest around the Rio Maullrin being submerged by the tides and consequently defoliated.

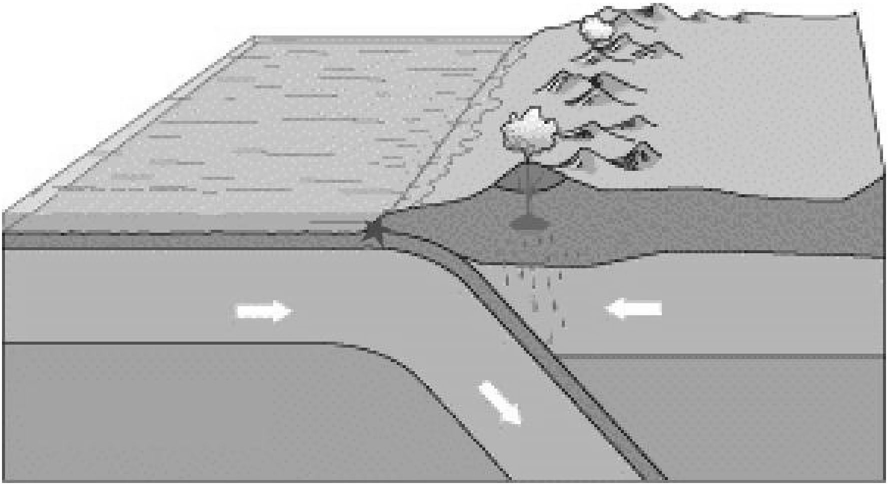

Figure 2.1 Earthquake causing the tsunami

Not only was the principal earthquake at 39.5.S, 74.5.W especially powerful, it generated a tsunami with an average run-up of 12.2 *m* and a maximal run-up on the adjacent Chilean coast of 25 *m*. Over the course of the next day, a number of tsunamis wreaked havoc upon the Pacific, taking the lives of more than 2000 people and causing millions of dollars in damages. The initial wave travelled between 670 and 740 *km/h*, with a wavelength of between 500-800 *km* and a height in the open ocean of only 40 *cm*. Borrowing an example, sitting in a boat in the Pacific, the tsunami wave would take between 45 minutes to an hour to pass one by while raising the boat by less than one centimetre per minute—hardly noticeable on the open sea. Nevertheless, the tsunami reached amplitudes of 7 *m* in Kamchatka and 10.7 *m* in Hilo, Hawaii, where it caused widespread destruction after travelling 10,000 *km* in just under 15 hours. The Chilean tsunami of 1960 had wavelengths in excess of 500 km and amplitudes of less than one metre while propagating over the Pacific Ocean, which, though the deepest of the worlds' oceans, has an average depth of only 4.3 *km*. These scales lend themselves to modelling with shallow-water long-wave theory, i.e. water depth is small compared to wavelength and depth is large compared to amplitude. We note also that the depth of open ocean across which the 1960 tsunami travelled is relatively uniform, and given that the rupture length exceeded the wavelength of the resulting tsunami, it is reasonable to assume the waves as two-dimensional; this is borne out (at least between Chile and Hawaii) by consulting travel time charts (Fig. 2.2).

The study of the propagation of tsunamis from their small disturbance at the sea level to the size they reach approaching the coast has involved the works of many scientists. It is clear that in order to predict accurately the appearance of a tsunami it is fundamental to build up a good model. From this point of view the most important tool in the context of water waves is soliton theory. Frequently in the literature it is stated that a tsunami is produced by a large enough soliton (Fig. 1.1). Solitons arise as special solutions of a widespread class of weakly nonlinear dispersive PDEs modelling water waves, such as the KdV or Camassa-Holm equation, representing to various degrees of accuracy approximations to the governing equations for water waves in the shallow water regime.

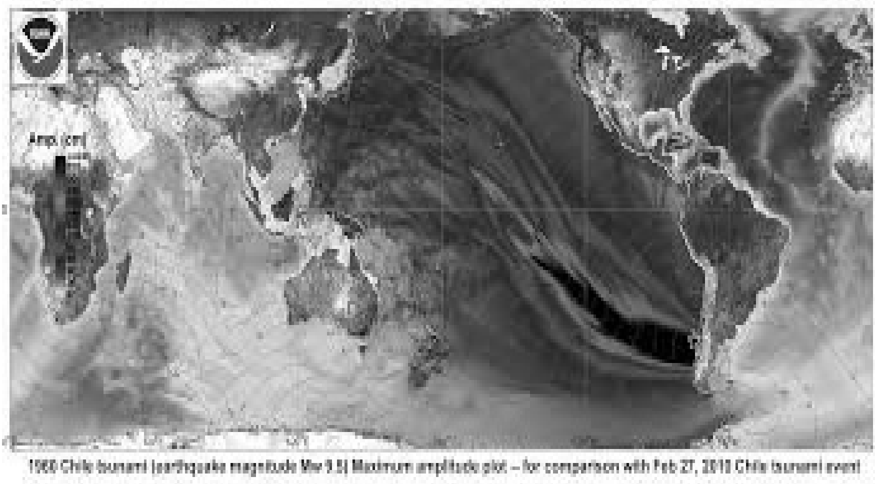

Figure 2.2 Chile tsunami

How is the tsunami initiated? The thrust of a mathematical approach is to examine how a wave, once initiated, moves, evolves and eventually becomes such a destructive force of nature. We aim to describe how an initial disturbance gives rise to a tsunami wave.

Mathematical modelling of tsunami waves

In order to derive the model equation of a tsunami wave we assume an initial disturbance of the form of a two-dimensional wave and we are interested in understanding the dynamics of the wave as it propagates across the ocean. Choose Cartesian coordinates (X, Y) with the Y-axis pointing vertically upwards, the X-axis being the direction of wave propagation, and with the origin located on the mean water level $Y = 0$. Let $\Phi(X, Y, T)$, $\Psi(X, Y, T)$ be the velocity field of the two-dimensional flow propagating in the X-direction over the flat bed $Y = -h$, and let $Y = H(X, T)$ be the water's free surface with the mean water level $Y = 0$. The equation of mass conservation

$$\Phi_X + \Psi_Y = 0$$

is a consequence of assuming constant density. Under the assumption of inviscid flow (which is realistic since experimental evidence confirms that the length scales associated with an adjustment of the velocity distribution due to laminar viscosity or turbulent mixing are long compared to the typical wavelengths) the equation of motion is Euler's equation:

$$\begin{cases} \Phi_T + \Phi\Phi_X + \Psi\Phi_Y = -\frac{1}{\rho} P_X \\ \Psi_T + \Phi\Psi_X + \Psi\Psi_Y = -\frac{1}{\rho} P_Y - g, \end{cases}$$

where P is the pressure, g is the constant acceleration of gravity and ρ is the constant density of the water. We also have the boundary conditions $P = P_{atm}$ on $Y = H(X, T)$, where P_{atm} is the (constant) atmospheric pressure at the water's free surface, $\Psi = H_T + \Phi H_X$ on $Y = H(X, T)$, and $\Psi = 0$ on $Y = -h$. These conditions express the fact that water particles cannot cross the free surface, respectively, the impermeable rigid bed, while $P = P_{atm}$ decouples the motion of the water from that of the air above it in the absence of surface tension; for wavelengths larger than a few *mm* (and in our case we deal with hundreds of *km*) the effects of surface tension are known to be negligible. We will consider irrotational flows with zero vorticity

$$\Phi_Y - \Psi_X = 0.$$

This hypothesis allows for uniform currents but neglects the effects of non-uniform currents in the fluid.

Finding exact solutions to the nonlinear governing equations of water waves is not possible even with the aid of the most advanced computers. In order to derive approximations to the governing equations it is useful to write them in non-dimensional form. We assume that the two-dimensional waves under investigation have acquired a certain pattern. We assume that the wave pattern under investigation represents a weakly irregular perturbation of a wave train in the sense that averages over suitable times/distances resemble a wave train. Since h is the average depth of the water, the non-dimensional Y_0 of Y should be $Y = hy$, which is to be understood as replacing the dimensional, physical variable Y by hy, where y is now a non-dimensional version of the original Y. The non-dimensional of the horizontal spatial variable is also obvious; if λ is an average of the typical wavelength of the wave, we set $X = \lambda x$. The corresponding non-dimensional time is $T = \frac{\lambda}{\sqrt{gh}}$t.

Then the governing equation for irrotational water wave equations in non- dimensional form is:

$$\begin{cases} \delta^2 U_{xx} + U_{yy} = 0 \ \ in\, \Gamma(t) \\ U_y = 0 \ \ \ on\, y = -1 \\ \xi_t + \varepsilon\, \xi_x U_x + \frac{\varepsilon}{\delta^2} U_y = 0\ on\, y = \varepsilon\xi \\ U_t + \frac{\varepsilon}{2} U_x^2 + \frac{\varepsilon}{2\delta^2} U_y^2 + \xi = 0\ on\, y = \varepsilon\xi, \end{cases}$$

where $x \mapsto \varepsilon\xi(x,t)$ is a parametrization on the free surface at time t, $\Gamma(t) = \{\, (x,y), -1 < y < \varepsilon\xi(x,t)\}$ is the fluid domain delimited above by the free surface and below by the flat bed $\{y = -1\}$, and where $U(.,.,t)\colon \Gamma \to \boldsymbol{R}$ is the velocity potential associated to the flow, so that the two-dimensional velocity field is given by (U_x, U_y).

An interesting phenomenon in water channels is the appearance of waves with a length much greater than the depth of the water. Korteweg and de Vries started the mathematical theory of this phenomenon and derived a model describing the unidirectional propagation of waves of the free surface of a shallow layer of water. This is the well-known KdV equation:

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0, t > 0, x \in \boldsymbol{R} \\ u(0,x) = u_0(x),\ x \in \boldsymbol{R}, \end{cases}$$

where u describes the free surface of the water. The beautiful structure behind the KdV equation initiated many mathematical investigations.

Recently, Camassa and Holm proposed a new model for the same phenomenon:

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, t > 0, x \in \boldsymbol{R} \\ u(0,x) = u_0(x), \ x \in \boldsymbol{R} \end{cases}$$

The variable $u(x,t)$ in the Camassa-Holm (CH) equation represents the fluid velocity at time t in the x direction in appropriate non-dimensional units (or, equivalently, the height of the water's free surface above a flat bottom). Unlike KdV, which is derived by asymptotic expansions in the equation of motion, CH is obtained by using asymptotic expansions directly in the Hamiltonian for Euler's equations in the shallow water regime. The novelty of Camassa and Holm's work was the physical derivation of the CH equation and the discovery that the equation has solitary waves (solitons) that retain their individuality under interaction and eventually emerge with their original shapes and speeds.

As an alternative model to KdV, Benjamin, Bona and Mahoney [9, 10, and 11] proposed the so-called BBM equation:

$$u_t + u_x + uu_x - u_{xxt} = 0, t > 0, x \in \boldsymbol{R}.$$

Numerical work of Bona, Pritchard and Scott shows that the solitary waves of the BBM equation are not solitons.

As noted by Whitham [118], it is intriguing to find mathematical equations including the phenomena of breaking and peaking, as well as criteria for the occurrence of each. He observed that solutions of the KdV equation do not break as physical water waves do. Whitham suggested replacing the KdV model with the nonlocal equation

$$u_t + uu_x + \boldsymbol{K} = 0, t > 0, x \in \boldsymbol{R},$$

for which he conjectured that breaking solutions exist. Here $\boldsymbol{K}$ is a Fourier operator with the symbol $k(\xi) = \sqrt{(\tanh \xi)/\xi}$. Whitham's conjecture was proved in Naumkin and Shishmarev, "Nonlinear Nonlocal Equations in the Theory of Waves," vol. 133, *Transl. Math.* (Rhode Island: Monographs, Providence, 1994). The numerical calculations carried out for the Whitham equation do not support any strong claim that soliton interaction can be expected.

On the other hand, Camassa, Holm and Hyman [19] show that the solitary waves have a discontinuity in the first derivative at their peak and that soliton interactions occur in the CH equation. The advantage of the new equation in comparison with the well-established models KdV, BBM and the Whitham equation is clear: the Camassa-Holm equation has peaked solitons, breaking waves, and permanent waves.

Discussion of Tsunami waves dynamics

Let us conclude this chapter with a brief discussion of the wave dynamics as the tsunami propagates towards the coast. The previous considerations show that from initiation until nearing the coastal region, a good approximation in the non-dimensional variables of tsunami waves is provided by the solutions of the corresponding model equation. In the original physical variables this means that up until the near-shore the wave profile remained unaltered propagating at constant speed $\sqrt{gh_0}$. The linear model breaks down when the tsunami waves enter the shallower water of the coastal regions and for an understanding of the tsunamis close to the shore the appropriate equations are those modelling the propagation of long water over variable depth. Before the waves reach the breaking state, their front steepens and dispersion, no matter how weak, becomes relevant. In this region faster wave fronts can catch up slower ones.

Let us take for example the tsunami of 2004 in the Indian Ocean [1]. For modelling purposes, outside of the Bay of Bengal the two-dimensional character of the tsunami waves can no longer be taken for granted since diffraction around islands and reflection from steep shores alter this feature considerably (see Fig. 2.3).

The earthquake that generated the tsunami changed the shape of the ocean floor by raising it by a few *m* to the west of the epicentre and lowering it to the east (over 100 *km* in the east-west direction and about 900 *km* in the north-south direction). The initial shape of the wave pattern that developed into the tsunami wave featured therefore to the west of the epicentre a wave of elevation followed by a wave of depression (that is, with water levels higher and respectively lower than normal), while to the east of the epicentre the initial wave profile consisted of a depression followed by an elevation. The fact that as the tsunami waves reached the shore in either direction, the shape of the initial disturbance (first a wave of elevation, then a wave of depression, respectively *vice-versa*) was not altered is of utmost importance in validating a theory for the wave dynamics on this occasion. This observation suggests that perhaps the shape of the tsunami waves remained approximately constant as they propagated across the Bay of Bengal. These clearly show a leading wave of elevation, followed by a wave of depression, a feature common both to the initial wave profile west of the epicentre and to the tsunami as it entered the coastal regions of India and Sri Lanka. These measurements also confirm another essential feature of tsunami waves: even though these waves reach large amplitudes due to the diminishing depth effect as they approach the shore (waves as high as 30 *m* were observed near the city Banda Aceh on the west coast of the northern tip of Sumatra about 160 *km* away from the epicentre of the earthquake), tsunami waves are barely noticeable at sea due to their small amplitude. Indeed, the satellite data show that the maximum amplitude of the waves, whether positive or negative with respect to the usual sea level, was less than 0.8 *m* over distances of more than 100 *km*.

Quantifying the dynamics of tsunami waves as they impact on coastal areas is a challenging mathematical and physical problem of the utmost importance. It is in this regime that dispersion (that was insignificant during the tsunami propagation at sea) starts to play an important role: before the waves reach the breaking state, their front steepens and dispersion, no matter how weak, becomes relevant [37]. In this region faster wave fronts can catch up slower ones (but they can never overtake them) as a manifestation of the "confluence of shocks" (see [118]) and can result in large amplitude wave fronts building up behind smaller ones [37].

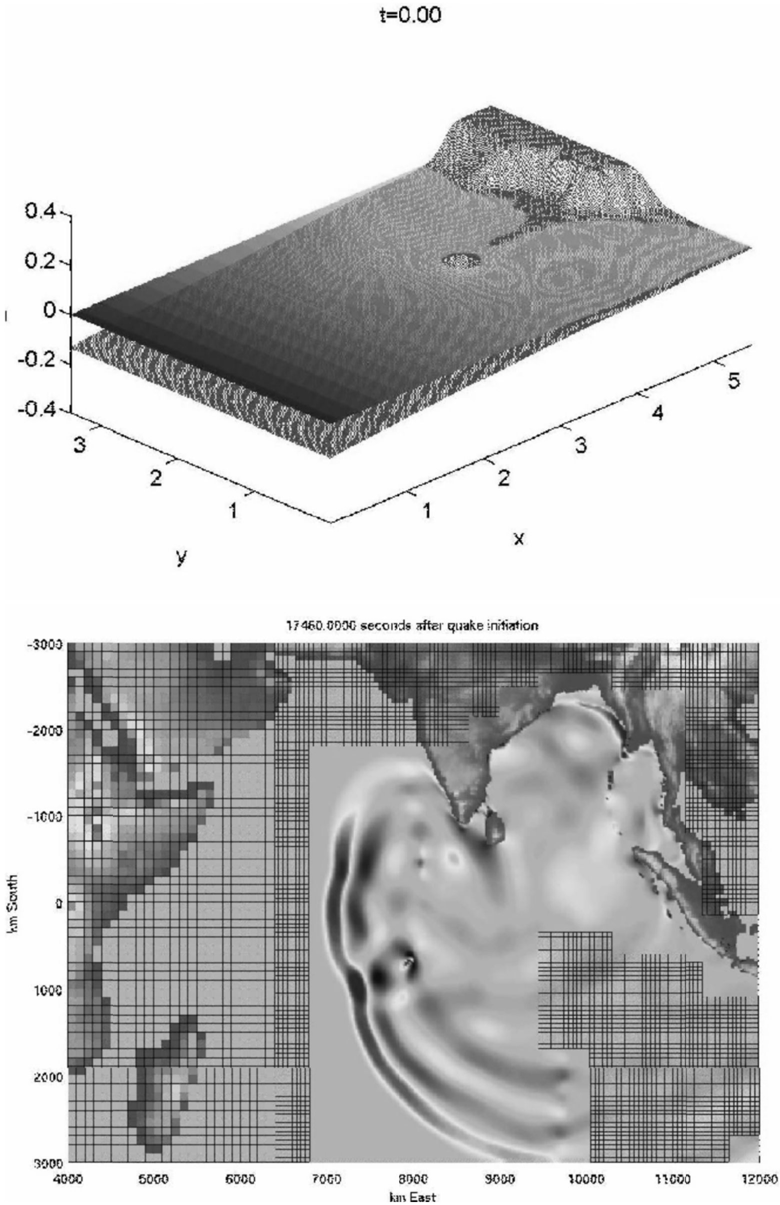

Figure 2.3 Tsunami in the Indian Ocean 2004

CHAPTER THREE

TRAVELLING WAVE SOLUTIONS OF SHALLOW WATER MODELS

Introduction

This Chapter deals with travelling wave solutions of shallow water waves. Camassa-Holm considered (see [18], [19]) a third order nonlinear PDE of two variables modelling the propagation of unidirectional irrotational shallow water waves over a flat bed, as well as water waves moving over an underlying shear flow. It is well known that the motion of inviscid fluid with a constant density is described by Euler's equations (a system of nonlinear PDE). In the special case of the motion of shallow water over a flat bottom the corresponding system was simplified by Green and Naghdi and related to an appropriate two-component first-order Camassa-Holm system. Another interesting system of nonlinear PDE is the viscoelastic generalization of Burger's equation. In the above-mentioned systems we are looking for travelling wave solutions and we are studying their profiles. To do this we use several results from the classical Analysis of ODE that enable us to give the geometrical picture and in several cases to express the solutions by the inverse of Legendre's elliptic functions. We are not going to discuss the physical explanation and interpretation and we shall concentrate only on the mathematical part of the investigations (see [83]).

As we know, one of the properties of the dispersive nonlinear evolution equations is that usually they possess steadily translating waves––the so-called travelling waves. By depending on specific boundary conditions on the wave's shape, for instance, in the case of water waves, these special states of motion can give rise to either solitary or periodic waves. Moreover, various nonlinear dispersive model equations are in sharp approximation to the governing equations for water waves. From these considerations, the problems about the stability of travelling waves and their existence as exact solutions of the dynamical equations are very important. The situation regarding periodic travelling waves is rather delicate. The stability and the existence of explicit formulas of these progressive wave trains have received little attention. A first study of these wave fronts is due to Benjamin in [10] with regard to the periodic steady solutions called cnoidal waves which were found initially by Korteweg and de Vries.

Travelling waves for Camassa-Holm type equations

It is classical that the soliton is a special solitary travelling wave that after a collision with another solution eventually emerges unscathed ([118], [10]). Solitons appear in the propagation of water waves or waves along a mass-spring chain. Travelling wave solutions of different classes of PDE are studied in many papers. We shall mention several of them only as they are closely connected with the content of this chapter: [18], [19], [36], [123], [84], and [94].

Camassa-Holm (see [18]) derived a shallow water equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{3.1}$$

and proved that it possesses a peaked solitary wave solution for $k = 0$. Degasperis and Procesi (see [46], [61]) proposed the following variant of (3.1):

$$m_t + um_x + bu_x m = c_0 u_x - \gamma u_{xxx}, \tag{3.2}$$

where $m = u - \alpha^2 u_{xx}$ and b, c_0, γ, α are constants, $\gamma \neq 0, \alpha \neq 0$.

Thus, (3.2) can be written as

$$u_t - c_0 u_x + (b+1)uu_x - \alpha^2(u_{xxt} + uu_{xxx} + bu_x u_{xx}) + \gamma u_{xxx} = 0. \tag{3.3}$$

The travelling wave solution of (3.3) is:

$$u(x,t) = \Phi(x - ct) = \Phi(\xi), c = const., \xi = x - ct.$$

Substituting u in (3.3) we get:

$$-(c + c_0)\Phi' + (b+1)\Phi\Phi' - \alpha^2(\Phi\Phi''' + b\Phi'\Phi'' + (\alpha^2 c + \gamma)\Phi''' = 0. \tag{3.4}$$

Remark 3.1.

Another possible generalization of (3.3) can be obtained by adding $u^r u_x = \Phi^r \Phi', r \in \boldsymbol{N}, r \geq 2$ *to the right-hand side of (3.3). Then* $\Phi^r \Phi' = \frac{1}{r+1}\frac{d}{d\,\xi}\Phi^{r+1}$;$\Phi'\Phi'' = \frac{1}{2}\frac{d}{d\xi}(\Phi')^2$,$\Phi\Phi''' = \frac{d}{d\xi}(\Phi\Phi''') - \frac{1}{2}\frac{d}{d\xi}(\Phi')^2$.

Integrating (3.4) with respect to ξ we have:

$$-(c+c_0)\Phi + \frac{1}{2}(b+1)\Phi^2 - (\alpha^2\Phi - \alpha^2 c\text{-}\gamma)\Phi'' - \frac{\alpha^2}{2}(b-1)(\Phi')^2 + g = 0, \qquad (3.5)$$

$g = const.$ Eventually, (3.5) contains the term $\frac{1}{r+1}\Phi^{r+1}$.

Our next step is to make in (3.5) the change $\Phi'_\xi = p(\Phi) \Rightarrow \Phi''_{\xi\xi} = \frac{1}{2}\frac{d}{d\Phi}(p^2)$. Put $p^2 = q \geq 0$. Then (3.5) takes the form

$$-(c+c_0)\Phi + \frac{1}{2}(b+1)\Phi^2 - \frac{1}{2}(\alpha^2\Phi - \alpha^2 c - \gamma)\frac{dq}{d\Phi} - \frac{\alpha^2}{2}(b-1)q + g = 0.$$

In a more general form

$$(\alpha^2\Phi - \alpha^2 c - \gamma)\frac{dq}{d\Phi} + \alpha^2(b-1)q + 2(c+c_0)\Phi - (b+1)\Phi^2 + \varepsilon\Phi^{r+1} - 2g = 0, \qquad (3.6)$$

where $\varepsilon^2 = \frac{2^2}{(r+1)^2}$ or $\varepsilon = 0$.

We shall concentrate on the case $\alpha^2\Phi - \alpha^2 c - \gamma \neq 0$, i.e. either $\Phi < \frac{\alpha^2 c+\gamma}{\alpha^2}$ or $\Phi > \frac{\alpha^2 c+\gamma}{\alpha^2}$.

The change of an independent variable Φ in (3.6): $\eta = \alpha^2\Phi - \alpha^2 c - \gamma$ leads to:

$$\eta\frac{dq}{d\eta} + (b-1)q + \frac{2(c+c_0)}{\alpha^4}(\eta + \alpha^2 c + \gamma) - (b+1)\frac{(\eta+\alpha^2 c+\gamma)^2}{\alpha^6} + \varepsilon\frac{(\eta+\alpha^2 c+\gamma)^{r+1}}{\alpha^{2(r+2)}} - \frac{2g}{\alpha^2} = 0 \qquad (3.7)$$